1. Let V be a subspace of $\mathbb{R}^5$ with basis $\{(2,2,1,0,0),(4,0,1,1,0),(1,0,-2,-2,0)\}$.

   (a) What is the dimension of V?

   (b) Use the Gram-Schmidt algorithm to find an orthonormal basis for V.

   (c) Find a matrix A with full row rank such that V is the row space of A. What is the rank of A? What are the dimensions of the four fundamental subspaces?

   (d) Is A^TA positive definite, positive semi-definite, or neither? Is AA^T positive definite, positive semi-definite, or neither?

   (e) Find a matrix B without full row rank such that V is the row space of B. What is the rank of B? What are the dimensions of the four fundamental subspaces?

Solution:

(a) Since V has a basis with three vectors, the dimension of V is 3.

(b) First we set

$$\mathbf{A} = \mathbf{a} = (2,2,1,0,0).$$

Next we set

$$\begin{aligned}\mathbf{B} &= \mathbf{b} - \frac{\mathbf{A}^T\mathbf{b}}{\mathbf{A}^T\mathbf{A}}\mathbf{A} \\ &= (4,0,1,1,0) - \frac{(2,2,1,0,0)^T(4,0,1,1,0)}{(2,2,1,0,0)^T(2,2,1,0,0)}(2,2,1,0,0) \\ &= (4,0,1,1,0) - \frac{9}{9}(2,2,1,0,0) \\ &= (2,-2,0,1,0)\end{aligned}$$

Finally, we set

$$\begin{aligned}\mathbf{C} &= \mathbf{c} - \frac{\mathbf{A}^T\mathbf{c}}{\mathbf{A}^T\mathbf{A}}\mathbf{A} - \frac{\mathbf{B}^T\mathbf{c}}{\mathbf{B}^T\mathbf{B}}\mathbf{B} \\ &= \begin{pmatrix}1\\0\\-2\\-2\\0\end{pmatrix} - \frac{(2,2,1,0,0)^T(1,0,-2,-2,0)}{(2,2,1,0,0)^T(2,2,1,0,0)}\begin{pmatrix}2\\2\\1\\0\\0\end{pmatrix} - \frac{(2,-2,0,1,0)^T(1,0,-2,-2,0)}{(2,-2,0,1,0)^T(2,-2,0,1,0)}\begin{pmatrix}2\\-2\\0\\1\\0\end{pmatrix} \\ &= (1,0,-2,-2,0) - \frac{0}{9}(2,2,1,0,0) - \frac{0}{9}(2,-2,0,1,0) \\ &= (1,0,-2,-2,0)\end{aligned}$$

We have the orthogonal basis

$$\{\mathbf{A},\mathbf{B},\mathbf{C}\} = \{(2,2,1,0,0),(2,-2,0,1,0),(1,0,-2,-2,0)\}.$$

To obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ we divide each vector by its length:

$$\mathbf{q}_1 = \frac{\mathbf{A}}{\|\mathbf{A}\|} = \frac{1}{3}(2, 2, 1, 0, 0).$$

$$\mathbf{q}_2 = \frac{\mathbf{B}}{\|\mathbf{B}\|} = \frac{1}{3}(2, -2, 0, 1, 0).$$

$$\mathbf{q}_3 = \frac{\mathbf{C}}{\|\mathbf{C}\|} = \frac{1}{3}(1, 0, -2, -2, 0).$$

(c) Either $A = \begin{pmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \\ 1 & 0 & -2 & -2 & 0 \end{pmatrix}$ or $A = \begin{pmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \mathbf{q}_3^T \end{pmatrix}$ would work.
Either way, the rank is 3, $\dim C(A^T) = 3$, $\dim N(A) = 2$, $\dim C(A) = 3$, and $\dim N(A^T) = 0$.

(d) Since A does not have independent columns, A^TA is positive semi-definite, but not positive definite. Since A^T has independent columns, and $(A^T)^T(A^T) = AA^T$, AA^T is positive definite.

(e) Any matrix B which has the same three rows of A, and an addition dependent row (linear combination of the other three) would work, for example

$$B = \begin{pmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \\ \mathbf{a}^T + \mathbf{b}^T \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \\ 1 & 0 & -2 & -2 & 0 \\ 6 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

In any case, the rank of B is 3, $\dim C(B^T) = 3$, $\dim N(B) = 2$, $\dim C(A) = 3$, and $\dim N(A^T) = 1$.

2. Suppose A is a diagonalizable matrix, and $A = X\Lambda X^{-1}$, where

$$X = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find $\det X$, $\det \Lambda$, and $\det A$.

(b) Find $\det X^5$, $\det \Lambda^5$, and $\det A^5$.

(c) List all eigenvectors and eigenvalues of A.

(d) Find two unit eigenvectors for A with eigenvalue 0.4.

(e) Is A invertible? Explain why or why not.

(f) Let $\mathbf{x} = -3(1, 1, 0, 1) + (1, 1, 0, 0) + 13(0, 1, 0, 0)$. What is $A^{17}\mathbf{x}$? (Round each component to the nearest integer)

(g) What is $X^{-1}(1, 1, 0, 1)$? (Do not compute X^{-1}.)

(h) Is A symmetric, positive semi-definite, positive definite, or none of these?

Solution:

(a)
- By using row operations, permutation formula, or cofactor formula, $\det X = -1$
- Since Λ is diagonal, $\det \Lambda$ is the product of the entries on the diagonal, i.e. $\det \Lambda = 0$
- Since the determinant is multiplicative, and $\det X^{-1} = 1/\det X = -1$,

$$\det A = \det(X\Lambda X^{-1}) = \det X \det \Lambda \det X^{-1} = (-1)(0)(-1) = 0$$

(b)
- $\det X^5 = (\det X)^5 = (-1)^5 = -1$
- $\det \Lambda^5 = (\det \Lambda)^5 = 0^5 = 0$
- $\det A^5 = (\det A)^5 = 0^5 = 0$

(c) The eigenvectors of A are the columns of X and the eigenvalues of A are the entries on the diagonal of Λ:
- $\mathbf{x}_1 = (1, 1, 0, 1)$, $\lambda_1 = 1$
- $\mathbf{x}_2 = (1, 0, 1, 0)$, $\lambda_2 = 0.5$
- $\mathbf{x}_3 = (1, 1, 0, 0)$, $\lambda_3 = 0.4$
- $\mathbf{x}_4 = (0, 1, 0, 0)$, $\lambda_4 = 0$

(d) Two unit eigenvectors for A with eigenvalue 0.4 are $(1/\sqrt{2}, 1/\sqrt{2}, 0, 0)$ and $(-1/\sqrt{2}, -1/\sqrt{2}, 0, 0)$

(e) Since $\det A = 0$, A is not invertible.

(f)

$$\begin{aligned} A^{17}\mathbf{x} &= A^{17}(-3(1, 1, 0, 0) + (1, 1, 0, 0) + 13(0, 1, 0, 0)) \\ &= -3A^{17}(1, 1, 0, 1) + A^{17}(1, 1, 0, 0) + 13A^{17}(0, 1, 0, 0)) \\ &= -3(1^{17})(1, 1, 0, 1) + (0.4)^{17}(1, 1, 0, 0) + 13(0^{17}(0, 1, 0, 0)) \\ &= -3(1, 1, 0, 1) + (0.4)^{17}(1, 1, 0, 0) \end{aligned}$$

Rounded to the nearest integer, $A^{17}\mathbf{x} = -3(1, 1, 0, 1) = (-3, -3, 0, -3)$.

(g) Since $X(1, 0, 0, 0) = (1, 1, 0, 1)$, $X^{-1}(1, 1, 0, 1) = (1, 0, 0, 0)$.

(h) Since the eigenvectors of A are not orthogonal, A is not symmetric.

3. Suppose A is a rank 2 matrix with singular values $\sigma_1 = 5$ and $\sigma_2 = 4$. Suppose the full format singular value decomposition for A is $A = U\Sigma V^T$ where U is 5×5 and V is 3×3, and the rank 2 format SVD is $A = U_2\Sigma_2 V_2^T$. Suppose $\mathbf{u}_i$ is the ith column of U and $\mathbf{v}_i$ is the ith column of V.

   (a) Write the matrix Σ.

   (b) How many rows and columns does A have?

   (c) How many rows and columns does U_2 have?

   (d) How many rows and columns does Σ_2 have?

   (e) How many rows and columns does V_2^T have?

   (f) What are the dimensions of the four fundamental subspaces of A? Find a basis for each in terms of the columns of U and V.

   (g) List all eigenvalues of A^TA.

   (h) Use $A = U\Sigma V^T$ to write a diagonalization for A^TA.

   (i) List all eigenvalues of AA^T.

   (j) What is $\det U^2$?

   (k) Write a singular value decomposition for A^T.

   (l) Use the SVD for A^T to find $A^T\mathbf{u}_2$?

   (m) Suppose $\mathbf{x} \in \mathbb{R}^3$ has $\|\mathbf{x}\| = 7$. What is $\|V^T\mathbf{x}\|$?

   (n) Suppose $\mathbf{x} \in \mathbb{R}^3$ has $\|\mathbf{x}\| = 7$. What is the greatest possible value of $\|A\mathbf{x}\|$? The smallest possible value?

Solution:

(a) $\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) A has 5 rows and 3 columns.

(c) U_2 has 5 rows and 2 columns.

(d) Σ_2 has 2 rows and 2 columns.

(e) V_2^T has 2 rows and 3 columns.

(f)
- $\dim C(A) = r = 2$, basis $= \{\mathbf{u}_1, \mathbf{u}_2\}$.
- $\dim N(A^T) = m - r = 3$, basis $= \{\mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$.
- $\dim C(A^T) = r = 2$, basis $= \{\mathbf{v}_1, \mathbf{v}_2\}$.
- $\dim N(A) = n - r = 1$, basis $= \{\mathbf{v}_3\}$.

(g) The eigenvalues of A^TA are 25, 16, 0.

(h) $A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T(U^TU)\Sigma V^T = V(\Sigma^T\Sigma)V^T$. I.e.

$$A^TA = V \begin{pmatrix} 25 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T.$$

(i) The eigenvalues of AA^T are 25, 16, 0, 0, 0.

(j) Since U is orthogonal, $\det U = \pm 1$, so $\det U^2 = (\det U)^2 = 1$.

(k) $A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$.

(l) $A^T\mathbf{u}_2 = V\Sigma^T(U^T\mathbf{u}_2) = V\Sigma^T\mathbf{e}_2 = V(4\mathbf{e}_2) = 4V\mathbf{e}_2 = 4\mathbf{v}_2$.

(m) Since V is orthogonal, $\|V^T\mathbf{x}\| = \|\mathbf{x}\| = 7$.
More details: Since V is orthogonal, $V^T = V^{-1}$, so

$$\|V^T\mathbf{x}\| = \sqrt{(V^T\mathbf{x})^T(V^T\mathbf{x})} = \sqrt{\mathbf{x}^T(VV^T)\mathbf{x}} = \sqrt{\mathbf{x}^T\mathbf{x}} = \|\mathbf{x}\| = 7.$$

(n) The greatest possible value of $\|A\mathbf{x}\|$ is $5 \cdot 7 = 35$ (achieved when $\mathbf{x}$ is a scalar multiple of $\mathbf{v}_1$). The smallest possible value of $\|A\mathbf{x}\|$ is $0 \cdot 7 = 0$ (achieved when $\mathbf{x}$ is a scalar multiple of $\mathbf{v}_3$).